



TITLE:

# Construction of wave operators for Hartree equations with a critical Hardy potential (Theory of Evolution Equation and Mathematical Analysis of Nonlinear Phenomena)

AUTHOR(S):

Suzuki, Toshiyuki

---

CITATION:

Suzuki, Toshiyuki. Construction of wave operators for Hartree equations with a critical Hardy potential (Theory of Evolution Equation and Mathematical Analysis of Nonlinear Phenomena). 数理解析研究所講究録 2018, 2090: 1-21

ISSUE DATE:

2018-09

URL:

<http://hdl.handle.net/2433/251617>

RIGHT:

# Construction of wave operators for Hartree equations with a critical Hardy potential

神奈川大学 工学部 数学教室 鈴木 敏行

Toshiyuki Suzuki

Department of Mathematics, Faculty of engineering, Kanagawa University

## 1. Introduction

In this article we consider the following Hartree equations with a Hardy potential:

$$(\mathbf{HE})_a \quad \begin{cases} i \frac{\partial u}{\partial t} = \left( -\Delta + \frac{a}{|x|^2} \right) u + u (K * |u|^2) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u(0) = u_0, \end{cases}$$

where  $i = \sqrt{-1}$ ,  $N \geq 3$  and

$$a \geq a(N) := -\frac{(N-2)^2}{4}.$$

$K * |u|^2$  is the usual convolution

$$(K * |u|^2)(x) := \int_{\mathbb{R}^N} K(x-y) |u(y)|^2 dy.$$

We suppose some conditions for  $K$  for analyzing  $(\mathbf{HE})_a$ :

(K1)  $K$  is real and even function, that is,  $K(-x) = K(x) \in \mathbb{R}$  a.a.  $x \in \mathbb{R}^N$ ;

(K2)  $K \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$  with  $q > N/4$  and  $q \geq 1$ ;

(K2a)  $K \in L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$  with  $q_1 \geq 1$  and  $N/4 < q_1 < q_2 \leq N/2$ ;

(K3)  $K_- := \max\{-K, 0\} \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$  with  $q > N/2$ .

(K3a)  $K \geq 0$  and  $\tilde{K} := 2K + x \cdot \nabla K \leq 0$ ;

(K4)  $\tilde{K} \in L^\infty(\mathbb{R}^N) + L^{\tilde{q}}(\mathbb{R}^N)$  with  $\tilde{q} > N/4$  and  $\tilde{q} \geq 1$ ;

(K4a)  $\tilde{K} \in L^{\tilde{q}_1}(\mathbb{R}^N) + L^{\tilde{q}_2}(\mathbb{R}^N)$  with  $\tilde{q}_1 \geq 1$  and  $N/4 < \tilde{q}_1 < \tilde{q}_2 < N/2$ .

Note that (K2a), (K3a), and (K4a) imply (K2), (K3), and (K4), respectively.

In general, semilinear (or nonlinear) Schrödinger equations is described strongly dispersive effects of waves, for example, propagation of signals in optical fibers. Especially,  $(\mathbf{HE})_a$  (without a linear potential term  $a|x|^{-2}u$ ) represents nonlocal interaction, for example, Hartree–Fock theory and WKB approximation for multi-body Schrödinger equation. On the other hand, the linear operator  $P_a := -\Delta + a|x|^{-2}$  arises from both physics and mathematics. In physical sides,  $P_a$  is concerned with quantum mechanics (Calogero–Moser system), wave propagation on conic manifolds, and combustion theory. On the one hand, in mathematical sides,  $P_a$  is concerned with scaling symmetry and the presense of threshold for the nonnegativity and selfadjointness. Since  $P_a[u(\lambda x)] = \lambda^2(P_a u)(\lambda x)$ , we see that  $(\mathbf{HE})_a$  can not be reduced to the case with  $|a|$  and  $\|u_0\|_{H^1}$  small enough. This implies that the term  $a|x|^{-2}$  represents non-negligible effect. Moreover, the restriction of  $a$  is affected by the Hardy inequality

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx.$$

Here the coefficient  $(N-2)^2/4$  is optimal.

Here one of keys for analyzing  $(\mathbf{HE})_a$  is the energy class  $\mathcal{D} := D((1 + P_a)^{1/2})$ . If  $a > a(N)$ , then  $\mathcal{D}$  is just equal to the usual Sobolev space  $H^1(\mathbb{R}^N)$ . If  $a = a(N)$ , then  $\mathcal{D}$  is a wider space than  $H^1(\mathbb{R}^N)$ . Thus we denote  $X^1(\mathbb{R}^N)$  as  $\mathcal{D}$  with  $a = a(N)$ . Note that  $H^1(\mathbb{R}^N) \subsetneq X^1(\mathbb{R}^N) \subsetneq H^s(\mathbb{R}^N)$  ( $s < 1$ ). In fact,

$$\|(-\Delta)^{s/2} f\|_{L^2} \leq \frac{\Gamma((N+2s)/4) \Gamma((1-s)/2)}{\Gamma((N-2s)/4) \Gamma((1+s)/2)} \|P_{a(N)}^{s/2} f\|_{L^2},$$

where  $\Gamma$  denotes the gamma functions (see Suzuki [11, Theorem 3.2]). We also denote  $\mathcal{D}^*$  as the conjugate space of  $\mathcal{D}$ . Thus  $H^1(\mathbb{R}^N)^* = H^{-1}(\mathbb{R}^N)$  and  $X^1(\mathbb{R}^N)^* = X^{-1}(\mathbb{R}^N)$ .

Local and global well-posedness for  $(\mathbf{HE})_a$  is proved in [9] for  $a > a(N)$  and [11] for  $a = a(N)$ .

**Proposition 1.1.** *Let  $a \geq a(N)$ . Assume **(K1)** and **(K2)**. Then for any  $u_0 \in \mathcal{D}$  there uniquely exists a local weak solution  $u \in C([-T, T]; \mathcal{D}) \cap C^1([-T, T]; \mathcal{D}^*)$  to  $(\mathbf{HE})_a$ . Moreover,  $u$  satisfies conservation laws:*

$$(1.1) \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E(u(t)) = E(u_0) \quad \forall t \in [-T, T],$$

where

$$\begin{aligned} E(\varphi) &:= \frac{1}{2} \|P_a^{1/2} \varphi\|_{L^2}^2 + \frac{1}{4} G[K](\varphi) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla \varphi|^2 + \frac{a}{|x|^2} |\varphi|^2 \right] dx + \frac{1}{4} \iint_{\mathbb{R}^{N+N}} K(x-y) |\varphi(x)|^2 |\varphi(y)|^2 dx dy. \end{aligned}$$

Furthermore, **(K3)** yields that the local weak solution of  $(\mathbf{HE})_a$  can be extended to the global weak solution  $u \in C(\mathbb{R}; \mathcal{D}) \cap C^1(\mathbb{R}; \mathcal{D}^*)$ .

If  $u_0 \in \mathcal{D}$  belongs also to  $D(|x|)$ , that is,  $|x|u_0 \in L^2(\mathbb{R}^N)$ , then the local weak solution  $u \in C([-T, T]; \mathcal{D}) \cap C^1([-T, T]; \mathcal{D}^*)$  to  $(\mathbf{HE})_a$  also belongs to  $C([-T, T]; D(|x|))$ . In fact, we see by a simple calculation that

$$\frac{d}{dt} \|xu(t)\|_{L^2}^2 = 4 \operatorname{Im} \int_{\mathbb{R}^N} \overline{xu(t, x)} \cdot \nabla u(t, x) dx.$$

Here the evaluation of  $\|xu(t)\|_{L^2}$  is available by assuming further **(K4)**. Actually, we call the identity about  $\|xu(t)\|_{L^2}$  the *virial identity*. The identity plays important roles in global analysis for  $(\mathbf{HE})_a$ :

$$(1.2) \quad \begin{aligned} \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 &= 8 \|P_a^{1/2} u(t)\|_{L^2}^2 \\ &\quad - 2 \iint_{\mathbb{R}^N} (x-y) \cdot \nabla K(x-y) |u(t, x)|^2 |u(t, y)|^2 dx dy \end{aligned}$$

(see [10, Section 3] for  $a > a(N)$  and [13, Section 3] for  $a = a(N)$ ). Applying (1.1) we obtain

$$(1.3) \quad \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 16 E(u_0) - 2 \iint_{\mathbb{R}^N} \tilde{K}(x-y) |u(t, x)|^2 |u(t, y)|^2 dx dy.$$

We can prove the finite time blowing up for  $(\mathbf{HE})_a$  via the virial identity (see [10, Theorem 1.2] for  $a > a(N)$  and [13, Theorem 4.1] for  $a = a(N)$ ).

**Proposition 1.2.** *Let  $a \geq a(N)$ . Assume (K1), (K2), (K4), and  $\tilde{K} \geq 0$ . Then for any  $u_0 \in \Sigma := \mathcal{D} \cap D(|x|)$  with  $E(u_0) < 0$  the local weak solution  $u \in C([-T, T]; \Sigma) \cap C^1([-T, T]; \mathcal{D}^*)$  to (HE) $_a$  cannot be extended globally in time  $t \in \mathbb{R}$ . More precisely, there exist  $T_1, T_2 > 0$  such that*

$$\lim_{t \rightarrow T_1 - 0} \|P_a^{1/2} u(t)\|_{L^2} = \infty, \quad \lim_{t \rightarrow -T_2 + 0} \|P_a^{1/2} u(t)\|_{L^2} = \infty.$$

We are interested in the asymptotic behavior of the global solutions to (HE) $_a$ . Note that the solutions are oscillating owing to the presence of  $i$  in the evolution equation (HE) $_a$  and the conservation laws (1.1). Thus we consider the existence of the following limits

$$u_{\pm} = \lim_{t \rightarrow \pm\infty} \exp(itP_a)u(t).$$

We say that (HE) $_a$  is *asymptotically free* in  $\Sigma$  if the limits exist for any solution to (HE) $_a$  with initial data belonging to  $\Sigma$ . The inverse mappings  $W_{\pm} : u_{\pm} \mapsto u_0$  may be also considered. The maps  $W_{\pm}$  are called *the wave operators* for (HE) $_a$ . To construct  $W_{\pm}$  we need to solve the following *final value* problems associated to (HE) $_a$ :

$$(FVP) \quad \begin{cases} i u_t = P_a u + u(K * |u|^2) & \text{in } (0, \infty), \\ \lim_{t \rightarrow \infty} \exp(itP_a)u(t) = u_+ & \text{strongly in } \Sigma. \end{cases}$$

In a way similar to Hayashi–Tsutsumi [4] we can apply the *pseudo-conformal transform* also to (HE) $_a$  and (FVP):

$$u(t, x) := (\mathcal{C}v)(t, x) = (it)^{-N/2} \exp\left(\frac{i|x|^2}{4t}\right) \overline{v\left(\frac{1}{t}, \frac{x}{t}\right)}.$$

By simple calculations we see that

$$\begin{aligned} \exp(-itP_a)D_{\nu} &= D_{\nu} \exp(-\nu^2 t P_a) \quad \forall t \in \mathbb{R}, \nu > 0, \\ \exp(-itP_a)M_b &= M_{b/(1+bt)} D_{1/(1+bt)} \exp\left(-\frac{it}{1+bt} P_a\right) \quad \forall t \in \mathbb{R}, b \in \mathbb{R} \text{ with } 1+bt > 0, \end{aligned}$$

where  $(D_{\nu}u)(x) := \nu^{N/2}u(\nu x)$  and  $(M_bu)(x) := \exp(ib|x|^2/4)u(x)$ . Thus we can rewrite  $(\mathcal{C}v)(t, x) = i^{-N/2} M_{1/t} D_{1/t} v(t^{-1}, x)$ . Note that we need to set  $\Sigma$  not as  $\mathcal{D}$  but as  $\mathcal{D} \cap D(|x|)$  (weighted energy space) so that the transform  $\mathcal{C}$  works well. In fact,

$$\|\nabla u(t)\|_{L^2} = \left\| \left( \frac{x}{2} + \frac{i}{t} \nabla \right) v(t^{-1}) \right\|_{L^2}, \quad \| |x|^{\omega} u(t) \|_{L^2} = |t|^{\omega} \| |x|^{\omega} v(t^{-1}) \|_{L^2} \quad t \in \mathbb{R}.$$

By applying  $D_{\nu}$  and  $M_b$ , we have

$$\exp(-i(1-t)P_a)u(t, x) = i^{-N/2} D_1 M_1 \overline{\exp(-i(1-t^{-1})P_a)v(t^{-1}, x)}.$$

Letting  $t \rightarrow \infty$  we see

$$(1.4) \quad \exp(-iP_a)u_+ = i^{-N/2} M_1 \overline{\exp(-iP_a)v(0, x)}.$$



Thus **(FVP)** is converted into the following *initial value* problems:

$$(IVP) \quad \begin{cases} i v_t = P_a v + t^{-2} v (D_{1/t} K * |v|^2), & \text{in } (0, \infty), \\ v(0) = v_+ := i^{-N/2} \exp(iP_a) e^{i|x|^2/4} \exp(iP_a) \bar{u}_+ & \text{in } \Sigma. \end{cases}$$

If we solve **(IVP)**, we can also solve **(FVP)**. Thus we can construct the wave operators for **(HE)<sub>a</sub>**. Suzuki [12] proved the scattering problems for **(HE)<sub>a</sub>** with the specified case by applying the contraction methods.

**Proposition 1.3.** *Let  $K(x) := |x|^{-\gamma}$ .*

- (i) *Assume that  $a \geq a(N)$  and  $1 < \gamma < \min\{N, 4\}$ . Then for every global solution  $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$  to **(HE)<sub>a</sub>** there exists  $u_+ \in L^2(\mathbb{R}^N)$  such that  $\exp(itP_a)u(t) \rightarrow u_+$  ( $t \rightarrow \infty$ ) strongly in  $L^2(\mathbb{R}^N)$ ;*
- (ii) *Assume either  $a > a(N)$  and  $1 < \gamma \leq 2$ , or  $a > (\gamma - 2)^2/4 + a(N)$  and  $2 < \gamma < \min\{N, 4\}$ . Then for every  $u_+ \in \Sigma$  there exists a global solution  $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$  to **(HE)<sub>a</sub>** such that  $\exp(itP_a)u(t) \rightarrow u_+$  ( $t \rightarrow \infty$ ) strongly in  $\Sigma$ .*

In this article we prove the scattering problems for **(HE)<sub>a</sub>** under more generalized cases via the energy methods.

**Theorem 1.4.** *Let  $a \geq a(N)$ . Assume **(K1)**, **(K2a)**, **(K3)**, and **(K4a)**. Then for any  $u_+ \in \Sigma$  there uniquely exists a solution  $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$  to **(FVP)**. Thus the wave operator  $W_+ : u_+ \mapsto u(0)$  is well-defined in  $\Sigma$ .*

On the contrary, we can show the asymptotic free in  $\Sigma$  of **(HE)<sub>a</sub>** in an almost similar way to Theorem 1.4.

**Theorem 1.5.** *Let  $a \geq a(N)$ . Assume that **(K1)**, **(K2a)**, **(K3a)**, and **(K4a)**. Then for any global solution  $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$  to **(HE)<sub>a</sub>** there exist the following limits*

$$\lim_{t \rightarrow \pm\infty} \exp(itP_a)u(t) = u_{\pm} \quad \text{strongly in } \Sigma.$$

Thus Theorems 1.4 and 1.5 imply that **(HE)<sub>a</sub>** is *asymptotically complete* in  $\Sigma$ , that is,  $W_{\pm}$  are bijective in  $\Sigma$  and the scattering operator  $\mathcal{S} := W_+^{-1} \circ W_-$  is well-defined.

This article is divided into 4 sections. In Section 2, we give the abstract theory related to **(IVP)**. In Section 3, we show Theorems 1.4 and 1.5 via the energy methods proposed in Section 2. In Section 4, we remark some comments about scattering problems for **(HE)<sub>a</sub>**.

## 2. Abstract theory for nonlinear Schrödinger equations

Let  $S$  be a nonnegative selfadjoint operator in a complex Hilbert space  $X$ . Put  $X_S := D((1 + S)^{1/2})$ . Then we have the usual triplet:  $X_S \subset X = X^* \subset X_S^*$ . Under this setting  $S$  can be extended to a nonnegative selfadjoint operator in  $X_S^*$  with domain  $X_S$ . Now we consider the abstract nonautonomous semilinear Schrödinger equations:

$$(ACP) \quad \begin{cases} i \frac{du}{dt} = Su + g(t, u) & t \in (-T, T), \\ u(0) = u_0 \in X_S. \end{cases}$$

$g(t, u)$  is a nonlinearity mapping from  $[-T, T] \times X_S$  to  $X_S^*$  under the following conditions. For simple notation we denote  $B_M := \{u \in X_S; \|u\|_{X_S} \leq M\}$ . Moreover,  $\varphi \in L^p(-T, T)$  ( $p > 1$ ) is a nonnegative function.

**(A1) Existence of energy functional of  $g$ :** for all  $t \in [-T, T]$ ,  $u \in X_S$ , and  $\varepsilon > 0$  there exists  $\delta = \delta(u, \varepsilon) > 0$  such that

$$|G(t, u + v) - G(t, v) - \operatorname{Re} \langle g(t, u), v \rangle_{X_S^*, X_S}| \leq \varepsilon \|v\|_{X_S} \quad \forall v \in B_\delta;$$

**(A2) Local Lipschitz continuity of  $u$ -variable:**

$$\|g(t, u) - g(t, v)\|_{X_S^*} \leq C(M) \|u - v\|_{X_S} \quad \forall t \in [-T, T], \forall u, v \in B_M;$$

**(A3) Hölder continuity of  $t$ -variable:**

$$\|g(t, u) - g(s, u)\|_{X_S^*} \leq C(M) \left| \int_s^t \varphi(\sigma) d\sigma \right| \quad \forall t, s \in [-T, T], \forall u \in B_M;$$

**(A4) Hölder-like continuity of energy functional:**

$$|G(t, u) - G(t, v)| \leq \delta + C_\delta(M) \|u - v\|_X \quad \forall \delta > 0, \forall t \in [-T, T], \forall u, v \in B_M;$$

**(A5) Partial differentiability of energy functional and Hölder-like continuity of  $u$ -variable:**

$$|G_t(t, u) - G_t(t, v)| \leq \varphi(t) [\delta + C_\delta(M) \|u - v\|_X] \quad \text{a.a. } t \in (-T, T), \forall u, v \in B_M;$$

**(A6) Gauge type condition for the conservation of charge:**

$$\operatorname{Re} \langle g(t, u), iu \rangle_{X_S^*, X_S} = 0 \quad \forall t \in [-T, T], \forall u \in X_S;$$

**(A7) Weak closedness condition:** let  $I \subset (-T, T)$  be an open interval and  $\{w_n\}_n \subset L^\infty(I; X_S)$ . Then

$$(2.1) \quad \begin{aligned} & \begin{cases} w_n(t) \rightarrow w(t) \ (n \rightarrow \infty) & \text{weakly in } X_S \text{ a.a. } t \in I, \\ g(t, w_n(t)) \rightarrow f(t) \ (n \rightarrow \infty) & \text{weakly}^* \text{ in } L^\infty(I; X_S^*) \end{cases} \\ & \Rightarrow \int_I \operatorname{Re} \langle f(t), i w(t) \rangle_{X_S^*, X_S} dt = \lim_{n \rightarrow \infty} \int_I \operatorname{Re} \langle g(t, w_n(t)), i w_n(t) \rangle_{X_S^*, X_S} dt. \end{aligned}$$

Moreover, if  $w_n(t) \rightarrow w(t)$  ( $n \rightarrow \infty$ ) strongly in  $X$  a.a.  $t \in I$ , then  $f(t) = g(t, w(t))$ ;

**(A8) Boundedness from below of  $G$ :** there exists  $\varepsilon > 0$  such that

$$G(t, u) \geq -[(1 - \varepsilon)/2] \|S^{1/2}u\|_X^2 - C(\|u\|_X) \quad \forall t \in [-T, T], \forall u \in X_S;$$

**(A9) Boundedness from below of  $G_t$ :** there exists  $\psi \in L^1(-T, T)$  with  $\psi(t) \geq 0$  such that

$$\operatorname{sgn}(t) G_t(t, u) \leq \psi(t) [\|S^{1/2}u\|_X^2 + C(\|u\|_X)] \quad \text{a.a. } t \in (-T, T), \forall u \in X_S.$$

If  $g$  maps unilaterally, from  $[0, T] \times X_S$  to  $X_S^*$ , then we consider the even extension:

$$g(t, u) := g(|t|, u), \quad G(t, u) := G(|t|, u) \quad \forall t \in [-T, T].$$

**Theorem 2.1 (Energy methods).** Assume (A1)–(A7). Then for any  $u_0 \in X_S$  there exists a local solution  $u \in C_w([-T_0, T_0]; X_S) \cap W^{1,\infty}(-T_0, T_0; X_S^*)$  to (ACP) with the following conservation laws

$$\|u(t)\|_X = \|u_0\|_X, \quad E(t, u(t)) - E(0, u_0) \leq \int_0^t G_t(s, u(s)) ds \quad \forall t \in [-T_0, T_0],$$

where  $E(t, u) := (1/2) \|S^{1/2}u\|_X^2 + G(t, u(t))$ . Moreover, assume further (A8) and (A9). Then the solution  $u$  can be extended globally in time  $t \in [-T, T]$ .

**Remark 2.1.** We need to prove uniqueness for (ACP) by another method. In fact, we verify the uniqueness for (HE)<sub>a</sub> and (IVP) by applying the Strichartz estimates (see Lemma 3.2). Here the uniqueness yields that the energy inequality of  $E$  is just an equality. Hence the solution  $u$  is strongly continuous:  $u \in C([-T_0, T_0]; X_S) \cap C^1([-T_0, T_0]; X_S^*)$ .

One of the keys for proving of Theorem 2.1 is the theory of nonautonomous semilinear evolution equation. Let  $X$  be a (complex-valued) Hilbert Banach space and  $A$  be a linear maximal monotone operator in  $X$ , that is,  $R(1 + A) = X$  and  $\operatorname{Re} \langle Au, u \rangle_X \geq 0$ . Then  $-A$  generates contraction  $C_0$ -semigroups  $\{e^{-tA}; t \geq 0\} \subset \mathcal{B}(X)$ , the family of bounded linear operators on  $X$ . Now we consider

$$(2.2) \quad \begin{cases} \frac{du}{dt} + Au + g_0(t, u) = 0 & \text{in } [0, T] \times X, \\ u(0) = u_0. \end{cases}$$

Assume that  $g_0$  satisfies

**(H1) Lipschitz continuity of  $g_0$  in  $u$ :** for all  $t \in [0, T]$ , and for any  $u, v \in X$  with  $\|u\|_X \leq M$  and  $\|v\|_X \leq M$

$$\|g_0(t, u) - g_0(t, v)\|_X \leq C(M)\|u - v\|_X;$$

**(H2) Hölder-like continuity of  $g_0$  in  $t$ :** there exists  $\varphi \in L^p(0, T)$  ( $p > 1$ ) with  $\varphi(t) \geq 0$  such that for all  $t, s \in [0, T]$  and for any  $u \in X$  with  $\|u\|_X \leq M$

$$\|g_0(t, u) - g_0(s, u)\|_X \leq C(M) \left| \int_s^t \varphi(\sigma) d\sigma \right|.$$

In a way similar to Cazenave–Haraux [3, Propositions 4.3.2 and 4.3.9] we can show the unique existence of solution to (2.2):

**Lemma 2.2.** Assume (H1) and (H2). Let  $u_0 \in D(A)$ . Then there uniquely exists  $u \in C([0, T_0]; D(A)) \cap C^1([0, T_0]; X)$  such that  $u$  is the local solution to (2.2). Here  $T_0 \in (0, T]$  is determined by  $\|u_0\|_X$ .

**Proof.** Unique existence of local solutions  $u \in C([0, T_0]; X)$  to (2.2) are followed by (H1) with a standard contraction argument for the integral equation related to (2.2):

$$u(t) = \Phi[u](t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}g_0(s, u(s)) ds.$$

It remains to show that the regularity of solution. Thus let  $u_0 \in D(A)$  and  $u \in C([0, T_0]; X)$  be a local unique solution to the above integral equation. Set  $h > 0$  sufficiently small and  $t \in [0, T_0 - h]$ . We divide  $u(t+h) - u(t) = \Phi[u](t+h) - \Phi[u](t)$  into  $I_0$ ,  $I_1$ ,  $I_2$ , and  $I_3$  as follows:

$$\begin{aligned} I_0 &:= e^{-(t+h)A}u_0 - e^{-tA}u_0, \\ I_1 &:= \int_0^t e^{-sA} [g(t+h-s, u(t+h-s)) - g(t+h-s, u(t-s))] ds, \\ I_2 &:= \int_0^t e^{-sA} [g(t+h-s, u(t-s)) - g(t-s, u(t-s))] ds, \\ I_3 &:= \int_0^h e^{-(t+h-s)A} g(s, u(s)) ds. \end{aligned}$$

First we see for  $I_0$  as a standard evaluation:

$$\|e^{-(t+h)A}u_0 - e^{-tA}u_0\|_X \leq h\|Au_0\|_X.$$

We can evaluate the norm of  $I_1$  by applying **(H1)**:

$$\begin{aligned} \|I_1\|_X &\leq \int_0^t \|g(t+h-s, u(t+h-s)) - g(t+h-s, u(t-s))\|_X ds \\ &\leq \int_0^t C(M) \|u(t+h-s) - u(t-s)\|_X ds = C(M) \int_0^t \|u(s+h) - u(s)\|_X ds. \end{aligned}$$

Next we consider  $I_2$ . Applying **(H2)** we have

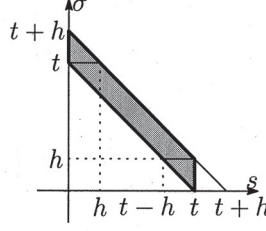
$$\begin{aligned} \|I_2\|_X &\leq \int_0^t \|g(t+h-s, u(t-s)) - g(t-s, u(t-s))\|_X ds \\ &\leq \int_0^t C(M) \left[ \int_{t-s}^{t+h-s} \varphi(\sigma) d\sigma \right] ds. \end{aligned}$$

Here the last integral is estimated by changing the order of integration (see Figure 1).

$$\begin{aligned} &\int_0^t \left[ \int_{t-s}^{t+h-s} \varphi(\sigma) d\sigma \right] ds \\ &= \int_0^h \left[ \int_{t-\sigma}^t \varphi(\sigma) ds \right] d\sigma + \int_h^t \left[ \int_{t-\sigma}^{t-\sigma+h} \varphi(\sigma) ds \right] d\sigma + \int_t^{t+h} \left[ \int_0^{t-\sigma+h} \varphi(\sigma) ds \right] d\sigma \\ &= \int_0^h \sigma \varphi(\sigma) d\sigma + \int_h^t h \varphi(\sigma) d\sigma + \int_t^{t+h} (t-\sigma+h) \varphi(\sigma) d\sigma \\ &\leq \int_0^h h \varphi(\sigma) d\sigma + \int_h^t h \varphi(\sigma) d\sigma + \int_t^{t+h} h \varphi(\sigma) d\sigma \leq h \int_0^T \varphi(\sigma) d\sigma. \end{aligned}$$

Thus we obtain

$$\|I_2\|_X \leq C(M)h \int_0^T \varphi(\sigma) d\sigma.$$

Figure 1: integration on  $I_2$ 

Next we evaluate  $I_3$  as follows:

$$\begin{aligned} \|I_3\|_X &\leq \int_0^h [\|g_0(s, u(s)) - g_0(s, 0)\|_X + \|g_0(s, 0)\|_X] ds \\ &\leq h[C(M)M + \|g(\cdot, 0)\|_{C([0, T]; X)}]. \end{aligned}$$

Combining the evaluation for  $I_j$  ( $j = 0, 1, 2, 3$ ), we obtain

$$\|u(t+h) - u(t)\|_X \leq C'(M)h + C(M) \int_0^t \|u(s+h) - u(s)\|_X ds,$$

where

$$C'(M) := C(M)M + \|g(\cdot, 0)\|_{C([0, T]; X)} + \int_0^T \varphi(\sigma) d\sigma.$$

The Gronwall lemma implies

$$\|u(t+h) - u(t)\|_X \leq C'(M) h e^{C(M)t}.$$

Since  $u$  is globally Lipschitz continuous in  $[0, T_0]$ ,  $u \in W^{1, \infty}(0, T_0; X)$ .

Next we show  $u \in C([0, T_0]; D(A)) \cap C^1([0, T_0]; X)$ . To derive this, it suffices to show that the nonlinear term  $g(t, u(t))$  belongs to  $W^{1, p}(0, T; X)$  ( $p > 1$ ):

$$\begin{aligned} &\|g(t+h, u(t+h)) - g(t, u(t))\|_X \\ &\leq \|g(t+h, u(t+h)) - g(t+h, u(t))\|_X + \|g(t+h, u(t)) - g(t, u(t))\|_X \\ &\leq C(M) \left| \int_t^{t+h} \varphi(\sigma) d\sigma \right| + C(M) \|u(t+h) - u(t)\|_X \\ &\leq \left| \int_t^{t+h} [C(M)\varphi(\sigma) + C(M)C'(M)e^{C(M)T}] d\sigma \right|. \end{aligned}$$

By virtue of Cazenave–Haraux [3, Proposition 4.1.6], we have proved the regularity of (local) weak solution to (2.2):  $u \in C([0, T]; D(A)) \cap C^1([0, T]; X)$ .  $\blacksquare$

Note that semilinear Schrödinger evolution equations can be solved backward and forward. Now we consider

$$(2.3) \quad \begin{cases} i \frac{du}{dt} = Su + g_0(t, u) & \text{in } [-T, T] \times X, \\ u(0) = u_0 \end{cases}$$



Assume that  $g_0$  satisfies **(H1)**, **(H2)** (with replacing  $[0, T]$  by  $[-T, T]$ ), and

**(H3) Existence of energy functional:** there exists  $G_0 \in C([-T, T] \times X; \mathbb{R})$  such that for all  $t \in [-T, T]$ ,  $u \in X$ , and  $\varepsilon > 0$  there exists  $\delta = \delta(u, \varepsilon) > 0$  such that

$$|G_0(t, u + v) - G_0(t, u) - \operatorname{Re} \langle g_0(t, u), v \rangle_X| \leq \varepsilon \|v\|_X \quad \forall v \in X \text{ with } \|v\|_X \leq \delta;$$

**(H4) Hölder-like continuity of  $G_{0t}$ :**  $G_0(t, u)$  is partially differentiable in  $t$  for any  $u \in X$ . Moreover, for any  $u \in X$  with  $\|u\|_X \leq M$

$$|G_{0t}(t, u) - G_{0t}(t, v)| \leq \varphi(t)[\delta + C_\delta(M)\|u - v\|_X] \quad \text{a.a. } t \in (-T, T);$$

**(H5) Gauge type condition:**

$$\operatorname{Re} \langle g_0(t, u), iu \rangle_X = 0 \quad \forall t \in [-T, T], \forall u \in X.$$

Apply Lemma 2.2 with letting  $A := \pm iS$  and replacing  $g_0(t, u)$  by  $\pm i g_0(\pm t, u)$  (double-sign corresponds). Thus **(H1)** and **(H2)** yield the unique existence of local solution  $u \in C([-T_0, T_0]; D(S)) \cap C^1([-T_0, T_0]; X)$  to (2.3). **(H3)**–**(H5)** imply the conservation laws:

$$(2.4) \quad \|u(t)\|_X = \|u_0\|_X, \quad E_0(t, u(t)) = E_0(0, u_0) + \int_0^t G_{0t}(s, u(s)) ds,$$

where  $E_0(t, u) := (1/2) \|S^{1/2}u\|_X^2 + G_0(t, u)$ . More precisely, **(H5)** implies the charge conservation (the former of (2.4)); **(H3)** and **(H4)** imply the energy conservation (the latter of (2.4)); By virtue of the conservation laws (2.4), the local solution can be extended globally in time  $t$ :  $u \in C([-T, T]; D(S)) \cap C^1([-T, T]; X)$ . Finally, arguments of denseness (see [2, Theorem 3.3.1]) follow the assertion.

**Lemma 2.3.** *Assume **(H1)**–**(H5)**. Then for any  $u_0 \in X_S$  there uniquely exists the global solution of (2.3)  $u \in C([-T, T]; X_S) \cap C^1([-T, T]; X_S^*)$ . Moreover,  $u$  satisfies the conservation laws (2.4).*

## 2.1. Outline of proof Theorem 2.1

Theorem 2.1 is proved in [14]. Now we give the outline of proof. In a way similar to [7] we divide into 5 steps as follows:

**Step 1.** Construct a global and approximated solution of **(ACP)**:

$$(\text{ACP})_\varepsilon \quad \begin{cases} i \frac{du_\varepsilon}{dt} = Su_\varepsilon + g_\varepsilon(t, u_\varepsilon) & t \in (-T, T), \\ u(0) = u_0 \in X_S, \end{cases}$$

where  $g_\varepsilon(t, u) := (1 + \varepsilon S)^{-1} g(t, (1 + \varepsilon S)^{-1}u)$ . Since  $g_\varepsilon$  maps from  $[-T, T] \times X$  to  $X$ , we can apply Lemma 2.3. **(H1)**–**(H5)** are verified by **(A2)**, **(A3)**, **(A1)**, **(A5)** and **(A6)**, respectively. Here  $u_\varepsilon \in C([-T, T]; X_S) \cap C^1([-T, T]; X_S^*)$  satisfies the following conservation laws:

$$\|u_\varepsilon(t)\|_X = \|u_0\|_X, \quad E_\varepsilon(t, u_\varepsilon(t)) = E_\varepsilon(0, u_0) + \int_0^t \partial_t G_\varepsilon(s, u_\varepsilon(s)) ds,$$

where  $G_\varepsilon(t, u) := G(t, (1 + \varepsilon S)^{-1}u)$  and

$$E_\varepsilon(t, u) := \frac{1}{2} \|(1 + S)^{1/2}u\|_X^2 + G_\varepsilon(t, u).$$

**Step 2.** Evaluate  $\|(1 + S)^{1/2}u_\varepsilon(t)\|_X$  uniformly in  $t \in [-T_M, T_M]$  and in  $\varepsilon > 0$ . This is the same way to [7]. To end this, we need to assume further **(A4)**.

**Step 3.** Confirm the weak convergence of  $(\mathbf{ACP})_\varepsilon$  to  $(\mathbf{ACP})$ . By virtue of Step 2, there exists the limit function  $u$  of  $u_\varepsilon$ , which satisfies

$$\begin{cases} i \frac{du}{dt} = Su + f(t) & t \in (-T_M, T_M), \\ u(0) = u_0 \in X_S. \end{cases}$$

Here  $f(t)$  is the weak\* limit of  $g_\varepsilon(t, u_\varepsilon(t))$  in  $L^\infty(-T_M, T_M; X_S^*)$ .

**Step 4.** Check the charge conservation and make a solution. By virtue of former half of **(A7)**, we obtain that

$$\operatorname{Re} \int_{-T_M}^{T_M} \langle f(t), i u(t) \rangle_{X_S^*, X_S} dt = 0.$$

This yields the charge conservation  $\|u(t)\|_X = \|u_0\|_X$ . Next, the charge conservation implies the strong convergence of  $u_\varepsilon$  in  $X$ . By virtue of latter half of **(A7)**, we see  $f(t) = g(t, u(t))$ . Hence we can show that the limit function  $u(t)$  is a just solution to **(ACP)**.

**Step 5.** Verify the energy pseudo-conservation. Weak convergence of  $u_\varepsilon(t)$  to  $u(t)$  in  $X_S$  and strong convergence of  $u_\varepsilon(t)$  to  $u(t)$  in  $X$  yield the energy pseudo-conservation.

### 3. Verifications of asymptotic completeness

#### 3.1. Proof of Theorem 1.4 (existence of wave operators)

To show Theorem 1.4, we prove the following assertion.

**Proposition 3.1.** *Let  $a \geq a(N)$ . Assume **(K1)**, **(K2a)**, and **(K4a)**. Then for any  $v_+ \in \mathcal{D}$  there uniquely exists a local weak solution  $v \in C([-T, T]; \mathcal{D}) \cap C^1([-T, T]; \mathcal{D}^*)$  to **(IVP)**. Moreover,  $v$  satisfies*

$$\|v(t)\|_{L^2} = \|v_+\|_{L^2},$$

$$E(t, v(t)) = E(0, v_+) - \int_0^t \frac{1}{4s^3} \left[ \iint_{\mathbb{R}^{N+N}} \tilde{K}\left(\frac{x-y}{|s|}\right) |v(s, x)|^2 |v(s, y)|^2 dx dy \right] ds,$$

where

$$E(t, u) := \frac{1}{2} \|P_a^{1/2}u\|_{L^2}^2 + \frac{1}{4t^2} \iint_{\mathbb{R}^{N+N}} K\left(\frac{x-y}{|t|}\right) |u(x)|^2 |u(y)|^2 dx dy.$$

Furthermore, if  $v_+ \in \Sigma = \mathcal{D} \cap D(|x|)$ , then  $v$  belongs also to  $C([-T, T]; \Sigma)$ .

To confirm Proposition 3.1 we check the uniqueness and the conditions **(A1)**–**(A7)** and apply Theorem 2.1. We define  $X := L^2(\mathbb{R}^N)$ ,  $S := P_a$ ,  $X_S := \mathcal{D}$ ,

$$(3.1) \quad g(t, v) := t^{-2}v(D_{1/|t|}K * |v|^2) = t^{-2}v \int_{\mathbb{R}^N} K\left(\frac{x-y}{|t|}\right) |v(y)|^2 dy,$$

$$(3.2) \quad G(t, v) := \frac{1}{4}G[t^{-2}D_{1/|t|}K](v) = \frac{1}{4t^2} \iint_{\mathbb{R}^{N+N}} K\left(\frac{x-y}{|t|}\right) |v(x)|^2 |v(y)|^2 dx dy.$$

$\mathcal{D}$  is the energy space related to  $P_a$ :

$$\|u\|_{\mathcal{D}} := \left[ \int_{\mathbb{R}^N} \left( |u|^2 + |\nabla u|^2 + \frac{a}{|x|^2} |u|^2 \right) dx \right]^{1/2}, \quad \mathcal{D} = \begin{cases} H^1(\mathbb{R}^N) & a > a(N), \\ X^1(\mathbb{R}^N) & a > a(N). \end{cases}$$

The Sobolev type embeddings are available:  $\mathcal{D} \subset L^r(\mathbb{R}^N)$  ( $2 \leq r < 2N/(N-2)$ ), more precisely,

$$(3.3) \quad \|u\|_{L^r} \leq c(r) \|u\|_{L^2}^{1-\theta} \|u\|_{\mathcal{D}}^{\theta} \leq c(r) \|u\|_{\mathcal{D}} \quad \forall u \in L^{r'}(\mathbb{R}^N),$$

where

$$\frac{1}{r} = \frac{1}{2} - \frac{\theta}{N}, \quad 0 < \theta < 1$$

(see Suzuki [11, Section 4] for  $a = a(N)$ ). Here we denote

$$\|u\|_{L^p(\Omega)} := \begin{cases} \left( \int_{\Omega} |u(x)|^p dx \right)^{1/p} & 1 \leq p < \infty, \\ \text{ess. sup } |u(x)| & p = \infty. \end{cases}$$

If  $\Omega = \mathbb{R}^N$ , then we omit to denote  $\mathbb{R}^N$ :  $\|u\|_{L^p} := \|u\|_{L^p(\mathbb{R}^N)}$ . Moreover, if  $\Omega \subset \mathbb{R}^N$  is a bounded open set with smooth boundary, then  $\mathcal{D} \subset L^r(\Omega)$  ( $2 \leq r < 2N/(N-2)$ ) is compact (The Rellich compactness lemma). On the other hand, since

$$\langle f, u \rangle_{\mathcal{D}^*, \mathcal{D}} = \int_{\mathbb{R}^N} \overline{f(x)} u(x) dx,$$

we see that

$$(3.4) \quad \|u\|_{\mathcal{D}^*} \leq c(r) \|u\|_{L^{r'}}, \quad \forall u \in L^{r'}(\mathbb{R}^N),$$

where  $r'$  is a Hölder conjugate of  $r \in [2, 2N/(N-2))$ :  $r' = r/(r-1)$ .

Also we see that

$$\|t^{-2} D_{1/|t|} K\|_{L^q} = |t|^{-2+N/q} \|K\|_{L^q}, \quad \partial_t [t^{-2} D_{1/|t|} K] = -t^{-3} D_{1/|t|} \tilde{K}.$$

We divide  $K$  and  $\tilde{K}$  into  $K_1 + K_2$  and  $\tilde{K}_1 + \tilde{K}_2$  so that  $K_j \in L^{q_j}(\mathbb{R}^N)$  and  $\tilde{K}_j \in L^{\tilde{q}_j}(\mathbb{R}^N)$  ( $j = 1, 2$ ). Note that **(K2a)** implies  $N/4 < q_1 < q_2 \leq N/2$  and **(K4a)** implies  $N/4 < \tilde{q}_1 < \tilde{q}_2 < N/2$ . The Young and the Hölder inequalities imply that

$$(3.5) \quad \left\| u_2(x) \int_{\mathbb{R}^N} K(x-y) u_3(y) u_4(y) dy \right\|_{L^{r'}} \leq \|K\|_{L^q} \|u_2\|_{L^r} \|u_3\|_{L^r} \|u_4\|_{L^r},$$

$$(3.6) \quad \left| \iint_{\mathbb{R}^{N+N}} K(x-y) u_1(x) u_2(x) u_3(y) u_4(y) dx dy \right| \\ \leq \|K\|_{L^q} \|u_1\|_{L^r} \|u_2\|_{L^r} \|u_3\|_{L^r} \|u_4\|_{L^r},$$

where  $r = 4q/(2q-1)$  and  $r' = 4q/(2q+1)$ .

**Verification of (A1).** Let  $u, v \in \mathcal{D}$ . Then we see from **(K1)** that

$$\begin{aligned}
 (3.7) \quad & G(t, u+v) - G(t, u) - \operatorname{Re} \langle g(t, u), v \rangle_{\mathcal{D}^*, \mathcal{D}} \\
 &= \frac{1}{4t^2} \iint_{\mathbb{R}^{N+N}} K\left(\frac{x-y}{|t|}\right) [|u+v(x)|^2 |u+v(y)|^2 - |u(x)|^2 |u(y)|^2] dx dy \\
 &\quad - \frac{1}{4t^2} \iint_{\mathbb{R}^{N+N}} K\left(\frac{x-y}{|t|}\right) [2 \operatorname{Re}(v(x) \overline{u(x)}) |u(y)|^2 + 2 \operatorname{Re}(v(y) \overline{u(y)}) |u(x)|^2] dx dy.
 \end{aligned}$$

Now let  $\alpha, \beta, \xi, \eta \in \mathbb{C}$ . Then we see that

$$\begin{aligned}
 (3.8) \quad & |\alpha + \xi|^2 |\beta + \eta|^2 - |\alpha|^2 |\beta|^2 - 2 |\beta|^2 \operatorname{Re}(\overline{\alpha} \xi) - 2 |\alpha|^2 \operatorname{Re}(\overline{\beta} \eta) \\
 &= 4 \operatorname{Re}(\overline{\alpha} \xi) \operatorname{Re}(\overline{\beta} \eta) + |\xi|^2 [|\beta|^2 + 2 \operatorname{Re}(\overline{\beta} \eta)] + |\eta|^2 [|\alpha|^2 + 2 \operatorname{Re}(\overline{\alpha} \xi)] + |\xi|^2 |\eta|^2.
 \end{aligned}$$

Put  $\alpha := u(x)$ ,  $\beta := u(y)$ ,  $\xi := v(x)$ ,  $\eta := v(y)$  in (3.8). It follows from (3.7) that

$$(3.9) \quad G(t, u+v) - G(t, u) - \operatorname{Re} \langle g(t, u), v \rangle_{\mathcal{D}^*, \mathcal{D}} = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
 I_1 &:= \frac{1}{t^2} \iint_{\mathbb{R}^{N+N}} K\left(\frac{x-y}{|t|}\right) \operatorname{Re}(\overline{u(x)} v(x)) \operatorname{Re}(\overline{u(y)} v(y)) dx dy, \\
 I_2 &:= \frac{1}{2t^2} \iint_{\mathbb{R}^{N+N}} K\left(\frac{x-y}{|t|}\right) |v(x)|^2 [|u(y)|^2 + 2 \operatorname{Re}(\overline{u(y)} v(y))] dx dy, \\
 I_3 &:= \frac{1}{4t^2} \iint_{\mathbb{R}^{N+N}} K\left(\frac{x-y}{|t|}\right) |v(x)|^2 |v(y)|^2 dx dy.
 \end{aligned}$$

We see from (3.6) and (3.3) that

$$\begin{aligned}
 (3.10) \quad |I_1| &\leq \sum_{j=1}^2 \left| \frac{1}{t^2} \iint_{\mathbb{R}^{N+N}} K_j\left(\frac{x-y}{|t|}\right) \operatorname{Re}(\overline{u(x)} v(x)) \operatorname{Re}(\overline{u(y)} v(y)) dx dy \right| \\
 &\leq \sum_{j=1}^2 t^{-2+N/q_j} \|K_j\|_{L^{q_j}} \|u\|_{L^{r_j}}^2 \|v\|_{L^{r_j}}^2 \leq \sum_{j=1}^2 c(r_j)^4 t^{-2+N/q_j} \|K_j\|_{L^{q_j}} \|u\|_{\mathcal{D}}^2 \|v\|_{\mathcal{D}}^2,
 \end{aligned}$$

where  $r_j = 4q_j/(2q_j - 1)$ . Note that  $2 < 2N/(N-1) \leq r_2 < r_1 < 2N/(N-2)$  by **(K2a)**.

In a way similar to  $I_1$ , we obtain

$$(3.11) \quad |I_2| \leq d_K(t) \|v\|_{\mathcal{D}}^2 [\|u\|_{\mathcal{D}}^2 + 2\|u\|_{\mathcal{D}} \|v\|_{\mathcal{D}}],$$

$$(3.12) \quad |I_3| \leq d_K(t) \|v\|_{\mathcal{D}}^4,$$

where

$$d_K(t) := \sum_{j=1}^2 c(r_j)^4 |t|^{-2+N/q_j} \|K_j\|_{L^{q_j}}.$$

Since  $-2 + N/q_1 > -2 + N/q_2 \geq 0$  by **(K2a)**, we see  $d_K(t) \leq d_K(T)$  for  $t \in [-T, T]$ .

(3.10), (3.11), and (3.12) imply that

$$\begin{aligned}
 & |G(t, u+v) - G(t, u) - \operatorname{Re} \langle g(t, u), v \rangle_{\mathcal{D}^*, \mathcal{D}}| \\
 &\leq d_K(T) \|v\|_{\mathcal{D}}^2 [6\|u\|_{\mathcal{D}}^2 + 4\|u\|_{\mathcal{D}} \|v\|_{\mathcal{D}} + \|v\|_{\mathcal{D}}^2] \quad \forall t \in [-T, T], \forall u, v \in \mathcal{D}.
 \end{aligned}$$

Let  $M > 0$  and  $\varepsilon > 0$ . Then we see that

$$\begin{aligned} |G(t, u + v) - G(t, u) - \operatorname{Re} \langle g(t, u), v \rangle_{\mathcal{D}^*, \mathcal{D}}| &\leq d_K(T) (6M^2 + 4M + 1) \|v\|_{\mathcal{D}}^2 \\ \forall t \in [-T, T], \forall u, v \in \mathcal{D} \text{ with } \|u\|_{\mathcal{D}} \leq M, \|v\|_{\mathcal{D}} \leq 1. \end{aligned}$$

Hence by setting  $\delta > 0$  as

$$\delta = \delta(u, \varepsilon) = 1 \wedge \frac{\varepsilon}{d_K(T) (6M^2 + 4M + 1)},$$

we conclude **(A1)**:

$$|G(t, u + v) - G(t, u) - \operatorname{Re} \langle g(t, u), v \rangle_{\mathcal{D}^*, \mathcal{D}}| \leq \varepsilon \|v\|_{\mathcal{D}} \quad \forall v \in \mathcal{D} \text{ with } \|v\|_{\mathcal{D}} \leq \delta.$$

**Verification of (A2).** First we define

$$g_j(t, u) := t^{-2} u \int_{\mathbb{R}^N} K_j \left( \frac{x - y}{|t|} \right) |u(y)|^2 dy$$

for  $j = 1, 2$ . Note that  $g(t, u) = g_1(t, u) + g_2(t, u)$ . Let  $u, v \in \mathcal{D}$ . Then we see that

$$g_j(t, u) - g_j(t, v) = u (t^{-2} D_{1/t} K_j) * [|u|^2 - |v|^2] + (u - v) (t^{-2} D_{1/t} K_j) * |v|^2.$$

Applying (3.5), we can calculate

$$\begin{aligned} (3.13) \quad & \|g_j(t, u) - g_j(t, v)\|_{L^{r'_j}} \\ & \leq \|u (t^{-2} D_{1/t} K_j) * [|u|^2 - |v|^2]\|_{L^{r'_j}} + \|(u - v) (t^{-2} D_{1/t} K_j) * |v|^2\|_{L^{r'_j}} \\ & \leq |t|^{-2+N/q_j} \|K_j\|_{L^{q_j}} \|u\|_{L^{r_j}} \|u + v\|_{L^{r_j}} \|u - v\|_{L^{r_j}} \\ & \quad + |t|^{-2+N/q_j} \|K_j\|_{L^{q_j}} \|u - v\|_{L^{r_j}} \|v\|_{L^{r_j}}^2 \\ & \leq |t|^{-2+N/q_j} \|K_j\|_{L^{q_j}} [\|u\|_{L^{r_j}}^2 + \|u\|_{L^{r_j}} \|v\|_{L^{r_j}} + \|v\|_{L^{r_j}}^2] \|u - v\|_{L^{r_j}}. \end{aligned}$$

Thus (3.3) and (3.4) yield that

$$\begin{aligned} (3.14) \quad & \|g(t, u) - g(t, v)\|_{\mathcal{D}^*} \leq \sum_{j=1}^2 c(r_j) \|g_j(t, u) - g_j(t, v)\|_{L^{r'_j}} \\ & \leq \sum_{j=1}^2 c(r_j) |t|^{-2+N/q_j} \|K_j\|_{L^{q_j}} [\|u\|_{L^{r_j}}^2 + \|u\|_{L^{r_j}} \|v\|_{L^{r_j}} + \|v\|_{L^{r_j}}^2] \|u - v\|_{L^{r_j}} \\ & \leq d_K(t) [\|u\|_{\mathcal{D}}^2 + \|u\|_{\mathcal{D}} \|v\|_{\mathcal{D}} + \|v\|_{\mathcal{D}}^2] \|u - v\|_{\mathcal{D}}. \end{aligned}$$

This implies **(A2)**.

**Verification of (A3).** We see

$$\begin{aligned} g(t, u) - g(s, u) &= u [t^{-2} D_{1/|t|} K - s^{-2} D_{1/|s|} K] * |u|^2 \\ &= u \left[ \int_s^t \frac{\partial}{\partial \sigma} (\sigma^{-2} D_{1/|\sigma|} K) d\sigma \right] * |u|^2 = u \left[ \int_s^t -\sigma^{-3} D_{1/|\sigma|} \tilde{K} d\sigma \right] * |u|^2 \\ &= - \sum_{j=1}^2 u \left[ \int_s^t \sigma^{-3} D_{1/|\sigma|} \tilde{K}_j d\sigma \right] * |u|^2. \end{aligned}$$



By virtue of (3.5) and (3.3), we obtain

$$\begin{aligned} & \left\| u \left[ \int_s^t \sigma^{-3} D_{1/|\sigma|} \tilde{K}_j d\sigma \right] * |u|^2 \right\|_{L^{\tilde{r}_j'}} \leq \left| \int_s^t \|\sigma^{-3} D_{1/|\sigma|} \tilde{K}_j\|_{L^{\tilde{q}_j}} d\sigma \right| \|u\|_{L^{\tilde{r}_j}}^3 \\ & \leq \left| \int_s^t |\sigma|^{-3+N/\tilde{q}_j} \|\tilde{K}_j\|_{L^{\tilde{q}_j}} d\sigma \right| \|u\|_{L^{\tilde{r}_j}}^3 \leq c(\tilde{r}_j)^3 \|\tilde{K}_j\|_{L^{\tilde{q}_j}} \left| \int_s^t |\sigma|^{-3+N/\tilde{q}_j} d\sigma \right| \|u\|_{L^{\tilde{r}_j}}^3, \end{aligned}$$

where  $\tilde{r}_j := 4\tilde{q}_j/(2\tilde{q}_j - 1) \in (2, 2N/(N - 2))$  by **(K4a)** and  $\tilde{r}_j' := 4\tilde{q}_j/(2\tilde{q}_j + 1)$ . Thus we see from (3.4) that

$$\|g(t, u) - g(s, u)\|_{\mathcal{D}^*} \leq \|u\|_{\mathcal{D}}^3 \sum_{j=1}^2 c(\tilde{r}_j)^3 \|\tilde{K}_j\|_{L^{\tilde{q}_j}} \left| \int_s^t |\sigma|^{-3+N/\tilde{q}_j} d\sigma \right|.$$

Since **(K4a)** implies  $-3 + N/\tilde{q}_1 > -3 + N/\tilde{q}_2 > -1$ , the integrands belong to  $L^p(-T, T)$  for some  $p > 1$ . This concludes **(A3)**.

**Verification of (A4).** First we define

$$G_j(t, u) := \frac{1}{4t^2} \iint_{\mathbb{R}^{N+N}} K_j\left(\frac{x-y}{|t|}\right) |u(x)|^2 |u(y)|^2 dx dy \quad (j = 1, 2).$$

Let  $u, v \in \mathcal{D}$ . Then we see from **(K1)** that

$$G_j(t, u) - G_j(t, v) = \frac{1}{4t^2} \iint_{\mathbb{R}^{N+N}} K_j\left(\frac{x-y}{|t|}\right) [|u(y)|^2 - |v(y)|^2] [|u(x)|^2 + |v(x)|^2] dx dy.$$

Applying (3.6) we have

$$|G_j(t, u) - G_j(t, v)| \leq t^{-2+N/q_j} \|K_j\|_{L^{q_j}} [\|u\|_{L^{r_j}}^2 + \|v\|_{L^{r_j}}^2] [\|u\|_{L^{r_j}} + \|v\|_{L^{r_j}}] \|u - v\|_{L^{r_j}}.$$

(3.3) yields for any  $t \in [-T, T]$  and for all  $u, v \in \mathcal{D}$  with  $\|u\|_{\mathcal{D}} \leq M, \|v\|_{\mathcal{D}} \leq M$

$$\begin{aligned} & |G_j(t, u) - G_j(t, v)| \\ & \leq c(r_j)^4 |t|^{-2+N/q_j} \|K_j\|_{L^{q_j}} [\|u\|_{\mathcal{D}}^2 + \|v\|_{\mathcal{D}}^2] [\|u\|_{\mathcal{D}} + \|v\|_{\mathcal{D}}] \|u - v\|_{L^2}^{\theta_j} \|u - v\|_{L^2}^{1-\theta_j} \\ & \leq c(r_j)^4 T^{-2+N/q_j} \|K_j\|_{L^{q_j}} 2^{2+\theta_j} M^{3+\theta_j} \|u - v\|_{L^2}^{1-\theta_j}, \end{aligned}$$

where  $\theta_j = N(2^{-1} - r_j^{-1}) = N/(4q_j) \in (0, 1)$  by **(K2a)**. Applying the Young inequality

$$y^{1-\theta} \leq \varepsilon + \theta \left( \frac{1-\theta}{\varepsilon} \right)^{(1-\theta)/\theta} y,$$

we see

$$|G_j(t, u) - G_j(t, v)| \leq \frac{1}{2} \delta + C_{j,\delta}(M) \|u - v\|_{L^2},$$

where  $C_{j,\delta}(M) := \theta_j [\delta^{-1} (1 - \theta_j) (2M)^{3+\theta_j} T^{-2+N/q_j} c(r_j)^4 \|K_j\|_{L^{q_j}}]^{(1-\theta_j)/\theta_j}$ . Since  $G(t, u) = G_1(t, u) + G_2(t, u)$ , we have confirmed **(A4)**.

**Verification of (A5).** By a standard argument of weak derivatives, we see that

$$G_t(t, u) = -\frac{1}{4t^3} \iint_{\mathbb{R}^{N+N}} \tilde{K}\left(\frac{x-y}{|t|}\right) |u(x)|^2 |u(y)|^2 dx dy.$$

Define

$$\tilde{G}_j(t, u) := \frac{1}{4t^3} \iint_{\mathbb{R}^{N+N}} \tilde{K}_j\left(\frac{x-y}{|t|}\right) |u(x)|^2 |u(y)|^2 dx dy \quad (j = 1, 2).$$

In a way similar to (A4) we see that for all  $u, v \in \mathcal{D}$  with  $\|u\|_{\mathcal{D}} \leq M, \|v\|_{\mathcal{D}} \leq M$

$$\begin{aligned} |\tilde{G}_j(t, u) - \tilde{G}_j(t, v)| &\leq c(\tilde{r}_j)^4 |t|^{-3+N/\tilde{q}_j} \|\tilde{K}_j\|_{L^{\tilde{q}_j}} 2^{2+\tilde{\theta}_j} M^{3+\tilde{\theta}_j} \|u - v\|_{L^2}^{1-\tilde{\theta}_j}, \\ &\leq |t|^{-3+N/\tilde{q}_j} \left[ \frac{1}{2} \delta + \tilde{C}_{j,\delta}(M) \|u - v\|_{L^2} \right], \end{aligned}$$

where  $\tilde{\theta}_j = N/(4\tilde{q}_j)$  and  $\tilde{C}_{j,\delta}(M) := \tilde{\theta}_j[\delta^{-1}(1 - \tilde{\theta}_j)(2M)^{3+\tilde{\theta}_j} c(\tilde{r}_j)^4 \|\tilde{K}_j\|_{L^{\tilde{q}_j}}]^{(1-\tilde{\theta}_j)/\tilde{\theta}_j}$ . Since  $-3 + N/\tilde{q}_1 > -3 + N/\tilde{q}_2 > -1$  by (K4a) and  $G_t(t, u) = -\tilde{G}_1(t, u) - \tilde{G}_2(t, u)$ , we obtain (A5).

**Verification of (A6).** (K1) implies (A6) by a simple calculation:

$$\begin{aligned} \operatorname{Re} \langle g(t, u), iu \rangle_{\mathcal{D}^*, \mathcal{D}} &= \operatorname{Re} \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} t^{-2} K\left(\frac{x-y}{|t|}\right) |u(y)|^2 \overline{u(x)} iu(x) dy \right] dx \\ &= \operatorname{Re} i \iint_{\mathbb{R}^{N+N}} t^{-2} K\left(\frac{x-y}{|t|}\right) |u(y)|^2 |u(x)|^2 dx dy = 0. \end{aligned}$$

**Verification of (A7).** Let  $I \subset \mathbb{R}$  be an open and bounded interval and assume that  $\{w_n\}_n$  is a sequence in  $L^\infty(I; \mathcal{D})$  satisfying

$$\begin{cases} w_n(t) \rightarrow w(t) \ (n \rightarrow \infty) & \text{weakly in } \mathcal{D} \text{ a.a. } t \in I, \\ g(t, w_n(t)) \rightarrow f(t) \ (n \rightarrow \infty) & \text{weakly}^* \text{ in } L^\infty(I; \mathcal{D}^*). \end{cases}$$

Since  $\{g_1(w_n)\}_n$  and  $\{g_2(w_n)\}_n$  are bounded in  $L^\infty(I; \mathcal{D}^*)$  and the Sobolev embeddings, there exist a subsequence  $\{w_{n(j)}\}_j$  of  $\{w_n\}_n$  and  $f_1, f_2 \in L^\infty(I; \mathcal{D}^*)$  such that

$$(3.15) \quad g_j(t, w_{n(m)}(t)) \rightarrow f_j(t) \ (m \rightarrow \infty) \text{ weakly}^* \text{ in } L^\infty(I; L^{r'_j}(\mathbb{R}^N)) \ (j = 1, 2).$$

To confirm (2.1) let  $\Omega \subset \mathbb{R}^N$  be an arbitrary bounded open subset with  $C^1$  boundary. Then

$$\begin{aligned} (3.16) \quad \langle f_j(t), w(t) \rangle_{L^{r'_j}(\Omega), L^{r_j}(\Omega)} &= \langle f_j(t) - g_j(t, w_{n(m)}(t)), w(t) \rangle_{L^{r'_j}(\Omega), L^{r_j}(\Omega)} \\ &\quad + \langle g_j(t, w_{n(m)}(t)), w(t) - w_{n(m)}(t) \rangle_{L^{r'_j}(\Omega), L^{r_j}(\Omega)} \\ &\quad + \langle g_j(t, w_{n(m)}(t)), w_{n(m)}(t) \rangle_{L^{r'_j}(\Omega), L^{r_j}(\Omega)} \\ &=: J_{j1}(t) + J_{j2}(t) + J_{j3}(t) \quad (j = 1, 2). \end{aligned}$$

The weak convergence (3.15) asserts that

$$(3.17) \quad \int_I J_{j1}(t) dt \rightarrow 0 \ (m \rightarrow \infty), \quad j = 1, 2.$$

Next we consider  $J_{j2}$ . The Rellich compactness lemma implies that  $w_{n(m)}(t) \rightarrow w(t)$  ( $m \rightarrow \infty$ ) strongly in  $L^{r_j}(\Omega)$  a.a.  $t \in I$ . It follows from the boundedness of  $\{g_j(w_{n(m)}(t))\}_m$

in  $L^{r_j}(\Omega)$  a.a.  $t \in I$  that  $I_{12}(t) \rightarrow 0$  ( $m \rightarrow \infty$ ) for a.a.  $t \in I$ . We see the boundedness of  $\{w_{n(m)}\}_m$  in  $L^\infty(-T, T; L^{r_j}(\Omega))$  and  $\{g_j(w_{n(m)})\}_m$  in  $L^\infty(I; L^{r'_j}(\Omega))$ . The dominated convergence lemma yields that

$$(3.18) \quad \int_I J_{j2}(t) dt \rightarrow 0 \quad (m \rightarrow \infty).$$

(K1) implies that  $\text{Im } J_{j3}(t) = 0$  a.a.  $t \in I$  ( $j = 1, 2$ ). Integrating (3.16) over  $I$  and using (3.17) and (3.18), we obtain

$$\text{Re} \int_I \langle f_j(t), i w(t) \rangle_{L^{r'_j}(\Omega), L^{r_j}(\Omega)} dt = 0.$$

Since  $\Omega$  is arbitrary and  $f = f_1 + f_2$ , (A6) implies (2.1):

$$\text{Re} \int_I \langle f(t), i w(t) \rangle_{\mathcal{D}^*, \mathcal{D}} dt = 0 = \lim_{n \rightarrow \infty} \text{Re} \int_I \langle g(t, w_n(t)), i w_n(t) \rangle_{\mathcal{D}^*, \mathcal{D}} dt.$$

Next we show that  $f(t) = g(t, w(t))$  by assuming further that  $w_n(t) \rightarrow w(t)$  ( $n \rightarrow \infty$ ) in  $L^2(\mathbb{R}^N)$  a.a.  $t \in I$ . Let  $M := \sup_n \|w_n\|_{L^\infty(I; \mathcal{D})}$ . It follows from (3.13), (3.3), and (3.4) that

$$\begin{aligned} \|g(t, w_n(t)) - g(t, w(t))\|_{\mathcal{D}^*} &\leq \sum_{j=1}^2 c(r_j)^4 6M^{2+\theta_j} |I|^{-2+N/q_j} \|K_j\|_{L^{q_j}} \|w_n(t) - w(t)\|_{L^2}^{1-\theta_j} \\ &\rightarrow 0 \quad (n \rightarrow \infty) \quad \text{a.a. } t \in I. \end{aligned}$$

Thus we see that  $g(t, w_n(t)) \rightarrow g(t, w(t))$  ( $n \rightarrow \infty$ ) strongly in  $L^\infty(I; \mathcal{D}^*)$  and (A7) is verified.

To show the uniqueness we apply the Strichartz estimates for  $\{e^{-itP_a}\}$  established by Burq, Planchon, Stalker and Tahvildar-Zadeh [1] (see also [6, Theorems 2.3 and 2.5]).

**Definition 3.1.** The pair  $(\tau, \rho)$  is called a *Schrödinger admissible pair* if

$$\frac{2}{\tau} + \frac{N}{\rho} = \frac{N}{2}, \quad \tau > 2, \quad \rho \geq 2.$$

**Lemma 3.2.** Let  $N \geq 3$ ,  $a \geq a(N)$  and  $(\tau, \rho)$  be a Schrödinger admissible pair. Then the following inequality holds:

$$(3.19) \quad \|\exp(-itP_a)\varphi\|_{L^\tau(\mathbb{R}; L^\rho)} \leq C_\tau \|\varphi\|_{L^2} \quad \forall \varphi \in L^2(\mathbb{R}^N).$$

Moreover, let  $(\tau_j, \rho_j)$  ( $j = 1, 2$ ) be Schrödinger admissible pairs. Then for all  $\Phi \in L^{\tau'_1}(\mathbb{R}; L^{\rho'_1}(\mathbb{R}^N))$

$$(3.20) \quad \left\| \int_0^t \exp(-i(t-s)P_a) \Phi(s, x) ds \right\|_{L^{\tau_2}(\mathbb{R}; L^{\rho_2})} \leq C_{\tau_1, \tau_2} \|\Phi\|_{L^{\tau'_1}(\mathbb{R}; L^{\rho'_1})}.$$

We exclude the endpoint  $(\tau, \rho) = (2, 2N/(N-2))$  from the Schrödinger admissible pair. Let  $a > a(N)$ . Burq, Planchon, Stalker, and Tahvildar-Zadeh [1, Theorem 3] confirmed (3.19) for the endpoint; Pierfelice [8, Theorem 2 in Section 3] confirmed (3.20) for the endpoint. On the other hand, Mizutani [5] showed that (3.19) and (3.20) for the endpoint are broken down for  $a = a(N)$ .

**Lemma 3.3.** *Let  $u_j$  ( $j = 1, 2$ ) be local weak solutions to (IVP) on  $I = (-T_1, T_2) \subset \mathbb{R}$  with initial values  $u_j(0) = u_0 \in \mathcal{D}$ . Then  $u_1(t) = u_2(t)$  on  $t \in I$ .*

**Proof.** Let  $u_j \in L^\infty(I; \mathcal{D})$  ( $j = 1, 2$ ) be local weak solutions to (IVP) on  $I$  with initial values  $u_j(0) = u_0$ . Then  $u_j$  ( $j = 1, 2$ ) satisfy the following integral equations:

$$u_j(t) = \exp(-itP_a)u_0 - i \int_0^t \exp(-i(t-s)P_a)g(s, u_j(s)) ds.$$

Therefore we see that  $v(t) := u_1(t) - u_2(t)$  satisfies

$$v(t) = -i \int_0^t \exp(-i(t-s)P_a)[g(s, u_1(s)) - g(s, u_2(s))] ds.$$

Here  $(8q_j/N, r_j)$  ( $j = 1, 2$ ) are Schrödinger admissible pairs. Applying (3.13) and the Strichartz estimates (3.20), we see that for every Schrödinger admissible pair  $(\tau, \rho)$ ,

$$\begin{aligned} (3.21) \quad & \left\| \int_0^t \exp(-i(t-s)P_a)[g_j(u_1(s)) - g_j(u_2(s))] ds \right\|_{L^\tau(I; L^\rho)} \\ & \leq C_{8q_j/N, \tau} \|g_j(u_1) - g_j(u_2)\|_{L^{(8q_j/N)'}(I; L^{r_j'})} \\ & \leq C_{8q_j/N, \tau} \|t^{-2} D_{1/t} K_j\|_{L^{(4q_j/N)'}(I; L^{q_j})} \\ & \quad \times [\|u_1\|_{L_t^\infty L^{r_j}}^2 + \|u_1\|_{L_t^\infty L^{r_j}} \|u_2\|_{L_t^\infty L^{r_j}} + \|u_2\|_{L_t^\infty L^{r_j}}^2] \|v\|_{L^{8q_j/N}(I; L^{r_j})}, \end{aligned}$$

where  $\|\cdot\|_{L_t^\infty L^p} := \|\cdot\|_{L^\infty(I; L^p)}$ . Putting  $(\tau, \rho) := (8q_j/N, r_j)$  ( $j = 1, 2$ ) in (3.21), we see that

$$\begin{aligned} (3.22) \quad & \|v\|_{L^{8q_1/N}(I; L^{r_1})} + \|v\|_{L^{8q_2/N}(I; L^{r_2})} \\ & \leq 3(C_{8q_1/N, 8q_1/N} + C_{8q_1/N, 8q_2/N} + C_{8q_2/N, 8q_1/N} + C_{8q_2/N, 8q_2/N}) M^2 \\ & \quad \times [\|t^{-2+N/q_1}\|_{L^{(4q_1/N)'}(I)} \|K_1\|_{L^{q_1}} + \|t^{-2+N/q_1}\|_{L^{(4q_2/N)'}(I)} \|K_2\|_{L^{q_2}}] \\ & \quad \times [\|v\|_{L^{8q_1/N}(I; L^{r_1})} + \|v\|_{L^{8q_2/N}(I; L^{r_2})}], \end{aligned}$$

where

$$M := \max_{j=1,2} \{\|u_j\|_{L^\infty(I; L^{r_1})}, \|u_j\|_{L^\infty(I; L^{r_2})}\}.$$

Since  $-2 + N/q_1 > -2 + N/q_2 \geq 0$  by (K2a), (3.22) yields

$$\|v\|_{L^{r(\gamma)}(I; L^{r_1})} + \|v\|_{L^\infty(I; L^{r_2})} \leq 0$$

for the interval  $I$  sufficiently small. Extending the interval step by step, we conclude the uniqueness on any interval  $I$ .  $\blacksquare$

Since (A1)–(A7) are verified and the uniqueness of local weak solutions for (IVP) is proved, Theorem 2.1 yields the unique existence of local weak solutions to (IVP). Moreover, in a way similar to (1.3) (for self-excited system (HE)<sub>a</sub>), the virial identity for (IVP) can be constructed owing to (K4a):

$$(3.23) \quad \frac{d}{dt} \|xv(t)\|_{L^2}^2 = 4 \operatorname{Im} \int_{\mathbb{R}^N} \overline{xv(t, x)} \cdot \nabla v(t, x) dx,$$

$$(3.24) \quad \frac{d^2}{dt^2} \|xv(t)\|_{L^2}^2 = 16 E(t, v(t)) - \frac{2}{t^2} \iint_{\mathbb{R}^N} \tilde{K}\left(\frac{x-y}{|t|}\right) |u(t, x)|^2 |u(t, y)|^2 dx dy.$$

Note that we do not differentiate the nonlinear term  $g(t, u) = u(t^{-2}D_{1/t}K * |u|^2)$  by  $t$  to deriving the virial identity. Thus Proposition 3.1 is fully proved.

To end this section, it remains to show the well-definedness of  $W_+$ . We have constructed the local solution  $v \in C([-T, T]; \Sigma) \cap C^1([-T, T]; \mathcal{D}^*)$  to **(IVP)**. Applying the pseudo-conformal transform  $\mathcal{C}$  we can define

$$u_{1/T} := (\mathcal{C}v)(1/T, x) = (-iT)^{N/2} e^{iT|x|^2/4} \overline{v(T, Tx)} \in \Sigma.$$

Proposition 1.1 implies that **(K1)**, **(K2)**, and **(K3)** admit a unique global weak solution  $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$  to

$$\begin{cases} i u_t = P_a u + u(K * |u|^2) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u(1/T) = u_{1/T} \in \Sigma. \end{cases}$$

The uniqueness of **(IVP)** implies that  $u(t, x) = (\mathcal{C}v)(t, x)$  on  $(1/T, \infty)$ . Thus there exists a unique global weak solution  $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$  to **(FVP)**. Hence Theorem 1.4 has been fully proved.

### 3.2. Proof of Theorem 1.5 (asymptotic free)

Now we show the asymptotic free of **(HE)<sub>a</sub>**. To end this, first we consider the global weak solution. Assume **(K1)**, **(K2)**, and **(K3)**. Let  $u_0 \in \Sigma$ . Then Proposition 1.1 implies that there uniquely exists a global weak solution  $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$  to **(HE)<sub>a</sub>**. Thus  $v = \mathcal{C}^{-1}u$  belongs to  $C((0, \infty); \Sigma) \cap C^1((0, \infty); \mathcal{D}^*)$  and satisfies

$$i \frac{\partial v}{\partial t} = P_a v + t^{-2} v (D_{1/t} K * |v|^2) \quad \text{on } (0, \infty).$$

To prove Theorem 1.5, In a view of 1.4, it is sufficient to show that  $v$  can be continuously extended to  $t = 0$ . Now we show the uniform boundedness of  $\|P_a^{1/2}v(t)\|_{L^2}$  in  $t \in (0, 1)$ . The energy conservation laws yields that

$$\begin{aligned} & \|(1 + P_a)^{1/2}v(t)\|_{L^2}^2 \\ &= \|(1 + P_a)^{1/2}v(1)\|_{L^2}^2 + 2G(1, v(1)) - 2G(t, v(t)) + 2 \int_1^t G_t(s, v(s)) ds. \end{aligned}$$

Here  $K \geq 0$  by **(K3a)** implies that

$$G(t, u) = \frac{1}{4t^2} \iint_{\mathbb{R}^{N+N}} K\left(\frac{x-y}{|t|}\right) |u(x)|^2 |u(y)|^2 dx dy \geq 0.$$

Moreover,  $\tilde{K} \leq 0$  by **(K3a)** implies that

$$G_t(t, u) = -\frac{1}{4t^3} \iint_{\mathbb{R}^{N+N}} \tilde{K}\left(\frac{x-y}{|t|}\right) |u(x)|^2 |u(y)|^2 dx dy \geq 0.$$

Thus we see the uniform boundedness:

$$\|(1 + P_a)^{1/2}v(t)\|_{L^2}^2 \leq \|(1 + P_a)^{1/2}v(1)\|_{L^2}^2 + 2G(1, v(1)) \quad t \in (0, 1).$$



On the other hand, [13, Lemma 3.1] implies

$$\left| \operatorname{Im} \int_{\mathbb{R}^N} x \bar{u} \cdot \nabla u \, dx \right| \leq \|xu\|_{L^2} \|(1 + P_a)^{1/2} u\|_{L^2}.$$

(3.23) ensures

$$\left| \frac{d}{dt} \|xv(t)\|_{L^2} \right| \leq 2\|(1 + P_a)^{1/2} v(t)\|_{L^2}.$$

The uniform boundedness of  $\|(1 + P_a)^{1/2} v(t)\|_{L^2}$  implies that there exists  $v_0 \in \Sigma$  such that  $v(t) \rightarrow v_0$  ( $t \rightarrow +0$ ) weakly in  $\Sigma$ . Here (K1), (K2a), (K3) and (K4a) yield the unique solution  $\tilde{v} \in C([0, \infty); \Sigma) \cap C^1([0, \infty); \mathcal{D}^*)$  to

$$\begin{cases} i \frac{\partial \tilde{v}}{\partial t} = P_a \tilde{v} + t^{-2} \tilde{v} (D_{1/t} K * |\tilde{v}|^2) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ \tilde{v}(0) = v_0 \in \Sigma. \end{cases}$$

The uniqueness on  $(1, \infty)$  implies that  $v(t) = \tilde{v}(t)$ . Since  $\tilde{v}$  is continuous in  $\Sigma$  at  $t = 0$ ,  $v$  can be continuously extended to  $t = 0$ .

**Remark 3.1.** Since  $g(t, u) = t^{-2} u D_{1/t} K * |u|^2$  satisfies  $\overline{g(t, u)} = g(t, \bar{u})$ , the wave operator  $W_-$  and the asymptotic free for  $t \rightarrow -\infty$  can be considered by comming down to  $W_+$  and  $t \rightarrow \infty$ , respectively. In fact,  $W_- u_- = \overline{W_+ \bar{u}_-}$  and

$$\lim_{t \rightarrow -\infty} \exp(itP_a)u(t) = \overline{\lim_{t \rightarrow +\infty} \exp(itP_a)\bar{u}(-t)}.$$

Note that if  $v$  is a unique solution to

$$\begin{cases} i v_t = P_a v + g(t, v) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ v(0) = v_0 \in \Sigma, \end{cases}$$

then  $w(t) := \overline{v(-t)}$  satisfies

$$\begin{cases} i w_t = P_a w + g(t, w) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ w(0) = \bar{v}_0 \in \Sigma. \end{cases}$$

## 4. Concluding remarks

### 4.1. Conditions for $K$

Conditions for the integrability of  $K$  can be relaxed.  $L^q(\mathbb{R}^N)$  can be replaced into the Lorentz space (or weak- $L^q$  space)  $L^{q,\infty}(\mathbb{R}^N)$ :

$$\|K\|_{L^{p,\infty}} := \sup_{z>0} z \mu(\{x \in \mathbb{R}^N; |K(x)| > z\})^{1/p} < \infty,$$

where  $\mu$  is the Lebesgue measure. For example, the usual Hartree kernel  $|x|^{-\gamma} \in L^{N/\gamma,\infty}(\mathbb{R}^N)$  ( $0 < \gamma < N$ ) and the Yukawa-type kernel  $e^{-\lambda|x|}|x|^{-\gamma} \in L^{N/\gamma,\infty}(\mathbb{R}^N)$  ( $0 < \gamma < N$ ,  $\lambda > 0$ ). Thus the scattering problems for usual Hartree equations can be solved.

**Corollary 4.1.** *Let  $a \geq a(N)$  and  $K(x) := e^{-\lambda|x|}|x|^{-\gamma}$  ( $2 < \gamma < \min\{N, 4\}$ ,  $\lambda \geq 0$ ). Note that  $\tilde{K} = (-\lambda|x| + 2 - \gamma)K$ .*

(i) *For any  $u_+ \in \Sigma$  there uniquely exists a solution  $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$  to (FVP). Thus the wave operators  $W_\pm : u_\pm \mapsto u(0)$  is well-defined in  $\Sigma$ .*

(ii) *For any global solution  $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$  to (HE) $_a$  with initial value  $u_0 \in \Sigma$  there exist the following limits*

$$\lim_{t \rightarrow \pm\infty} \exp(itP_a)u(t) = u_\pm \quad \text{strongly in } \Sigma.$$

On the other hand, nonnegativity of  $K$  can be also relaxed:

$$\iint_{\mathbb{R}^{N+N}} K(x-y)\varphi(x)\varphi(y) dx dy \geq 0$$

for any measurable and nonnegative function  $\varphi$ . For example,  $\mathcal{F}K(\xi) \geq 0$  a.e. on  $\mathbb{R}^N$ , where  $\mathcal{F}$  is the Fourier transform. In fact, it follows from the Plancherel lemma and the Parseval identities that

$$\begin{aligned} \iint_{\mathbb{R}^{N+N}} K(x-y)\varphi(x)\varphi(y) dx dy &= \int_{\mathbb{R}^N} (K * \varphi)(x) \overline{\varphi(x)} dx \\ &= \int_{\mathbb{R}^N} \mathcal{F}(K * \varphi)(\xi) \overline{\mathcal{F}\varphi(\xi)} d\xi = \int_{\mathbb{R}^N} (2\pi)^{N/2} \mathcal{F}K(\xi) \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\varphi(\xi)} d\xi \\ &= \int_{\mathbb{R}^N} (2\pi)^{N/2} \mathcal{F}K(\xi) |\mathcal{F}\varphi(\xi)|^2 d\xi \geq 0. \end{aligned}$$

## 4.2. Abstract theory

Lemma 2.3 can be generalized for applying the systems of nonautonomous semilinear Schrödinger evolution equations. Let  $B : X_S^* \rightarrow X_S^*$  be a bounded linear operator with the following conditions:

- $BSu = SBu$  for  $u \in X_S$ ;
- $B$  is bounded and symmetric operator in  $X$ ;
- $B$  is coercive in  $X$ : there exists  $\varepsilon > 0$  such that  $\operatorname{Re}\langle Bu, u \rangle_X \geq \varepsilon \|u\|_X^2$ .

By using  $B$ , (H5) is replaced with (H5a):

$$\operatorname{Re}\langle g_0(t, u), iBu \rangle_X = 0 \quad \forall t \in [-T, T], \forall u \in X.$$

**Lemma 4.2.** *Assume (H1)–(H4) and (H5a). Then for any  $u_0 \in X_S$  there uniquely exists the global solution of (2.3)  $u \in C([-T, T]; X_S) \cap C^1([-T, T]; X_S^*)$ . Moreover,  $u$  satisfies the conservation laws*

$$\operatorname{Re}\langle Bu(t), u(t) \rangle_X = \operatorname{Re}\langle Bu_0, u_0 \rangle_X, \quad E_0(t, u(t)) = E_0(0, u_0) + \int_0^t G_{0t}(s, u(s)) ds.$$

Also, we can generalize Theorem 2.1 in a way similar to Section 2.1.

## References

- [1] N. Burq, F. Planchon, J. Stalker, A.S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*, J. Funct. Anal. **203** (2003), 519–549.
- [2] T. Cazenave, “Semilinear Schrödinger Equations,” Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, AMS, New York, 2003.
- [3] T. Cazenave, A. Haraux, “An Introduction to Semilinear Evolution Equations,” Oxford Lecture Series in Mathematics and its Applications, 13. The Clarendon Press, Oxford University Press, New York, 1998.
- [4] N. Hayashi, Y. Tsutsumi, *Scattering theory for Hartree type equations*, Ann. Inst. Henri Poincaré **46** (1987), 187–213.
- [5] H. Mizutani, *Remarks on endpoint Strichartz estimates for Schrödinger equations with the critical inverse-square potential*. J. Differ. Equ. **263** (2017), 3832–3853.
- [6] N. Okazawa, T. Suzuki, T. Yokota, *Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials*, Appl. Anal. **91** (2012), 1605–1629.
- [7] N. Okazawa, T. Suzuki, T. Yokota, *Energy methods for abstract nonlinear Schrödinger equations*, Evol. Equ. Control Theory **1** (2012), 337–354.
- [8] V. Pierfelice, *Weighted Strichartz estimates for the Schrödinger and wave equations on Damek-Ricci spaces*, Math. Z. **260** (2008), 377–392.
- [9] T. Suzuki, *Energy methods for Hartree type equation with inverse-square potentials*, Evol. Equ. Control Theory, **2** (2013), 531–542.
- [10] T. Suzuki, *Blowup of nonlinear Schrödinger equations with inverse-square potentials*, Differ. Equ. Appl., **6** (2014), 309–333.
- [11] T. Suzuki, *Solvability of nonlinear Schrödinger equations with some critical singular potential via generalized Hardy-Rellich inequalities*, Funkcial. Ekvac., **59** (2016), 1–34.
- [12] T. Suzuki, *Scattering theory for Hartree equations with inverse-square potentials*, Appl. Anal., **96** (2017), 2032–2043.
- [13] T. Suzuki, *Virial identities for nonlinear Schrödinger equations with an inverse-square potential of critical coefficient*, Differ. Equ. Appl., **9** (2017), 327–352 .
- [14] T. Suzuki, *Scattering theory for semilinear Schrödinger equations with an inverse-square potential via energy methods*, preprint.